Mean Value Theorem

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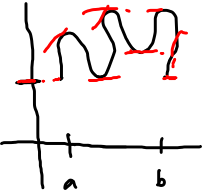

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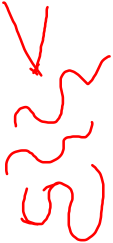
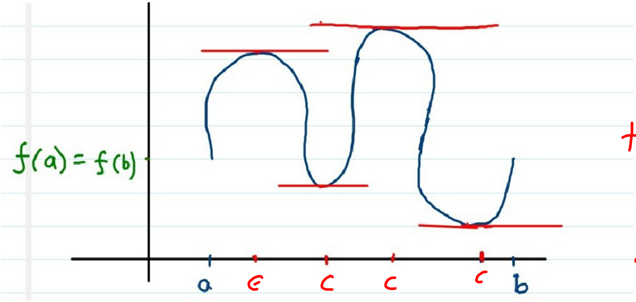
- What do you observe ? Write in terms of the slopes of the tangents
- Formulate the Statement of the theorem

smooth

= Differential





$f'(c) = 0$

tgt is parallel to x-axis

Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) and if $f(a) = f(b)$, then there is a $c \in (a, b)$ such that $f'(c) = 0$.

Proof.

Recall that a continuous function on a closed and bounded interval is bounded and attains its bound.

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Case 1: Suppose $a = x_0$ and $b = y_0$. In this case, the condition $f(a) = f(b)$ implies that $\sup f = \inf f$ on $[a, b]$ so that f is a constant and $f'(x) = 0, \ \forall x \in [a, b]$.

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Case 2: Suppose $x_0 \in (a, b)$.

x_0

$f'(x_0) \geq 0$

$f'(x_0) \leq 0$

$\Rightarrow f'(x_0) = 0$

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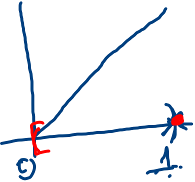
Case 2: Suppose $x_0 \in (a, b)$. In this case, $f(x_0)$ being $\sup\{f(x) : a \leq x \leq b\}$, we have $f'(x_0) = 0$. $\square$

The Conclusion of the Rolle's theorem does not hold if any one of the conditions is dropped.



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- Consider the function defined by $f(x) = x$ on $[0,1)$ and $f(1) = 0$ Then f is differentiable on $(0,1)$ and $f(0) = f(1)$, but $f'(x) = 1, \ \forall x \in (0,1)$.

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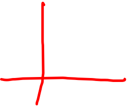
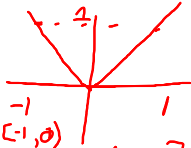
- The function $f(x) = \|x\|$ is continuous on $[-1, 1]$ and $f(1) = f(-1)$, but there is no $c \in (-1, 1)$ such that $f'(c) = 0$.

$|x|$



$\underline{(-1, 1)}$



$[-1, 0)$ $f' = -1$

$(0, 1]$ $f' = 1$

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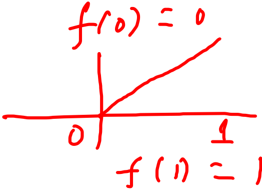

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- The function $f(x) = \|x\|$ is continuous on $[-1, 1]$ and $f(1) = f(-1)$, but there is no $c \in (-1, 1)$ such that $f'(c) = 0$. Here, f is not differentiable at $x = 0$.
- The function $f(x) = x$ is continuous on $[0, 1]$ and differentaible on $(0, 1)$ but $f'(x) = 1, \ \forall x \in (0, 1)$. Here $f(0) \neq f(1)$.

Suppose f has 2 roots

Cubic $\Rightarrow$ It has a real root

$f(a) = f(b) = 0$

$f'(c) = 0$

1. Prove that a cubic polynomial of the form $f(x) = x^3 + px + q, \quad p > 0$ has a unique real root.

$f'(x) = 3x^2 + p$

$\therefore f'(c) = 0$

$\Rightarrow 3c^2 + p = 0$

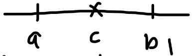
$3c^2 = -p$

contradiction

Between 2 roots of f there is a root of f'

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = p(x)$$

$p'(x)$ is a poly of deg $(n-1)$



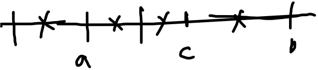
$f(a) = 0$

$f'(c) = 0$

$f(b) = 0$

1. Prove that a cubic polynomial of the form $x^3 + px + q, \quad p > 0$ has a unique real root.
2. Prove that a polynomial of degree n with real coefficients have at most n roots in $\mathbb{R}$.

– Apply induction.



deg 2

1. Prove that a cubic polynomial of the form $x^3 + px + q, \quad p > 0$ has a unique real root.
2. Prove that a polynomial of degree n with real coefficients have at most n roots in $\mathbb{R}$.
3. Prove that the equation $2x - 1 = \sin x$ has exactly one solution.

$f(x) = \sin x - 2x + 1$

$f' = \cos x - 2$

$\nexists \, c$ s.t $f'(c) = 0$

we cannot have $f(a) = f(b)$

Lagrange's Mean Value Theorem

Theorem

If $f : [a, b] \to \mathbb{R}$ *is continuous on* $[a, b]$ *and differentiable on* (a, b) *then there is a* $c \in (a, b)$ *such that* $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Lagrange's Mean Value Theorem

Construction of auxillary f^n in the proof of Lagrange's thm

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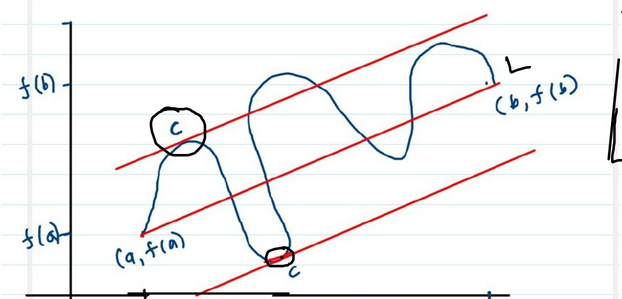

slope of L is

$$\boxed{\frac{f(b)-f(a)}{b-a}}$$

$g'(x)$
$= f'(x)$
$- \left(\frac{f(b) - f(a)}{b - a} \right)$

Proof.

Define $g(x) = f(x) - (\frac{f(b)-f(a)}{b-a})x$

Proof.

Define $g(x) = f(x) - (\frac{f(b)-f(a)}{b-a})x$

Then observe that g satisfies the conditions of the Rolle's theorem.

Check that $g(a) = g(b)$

Proof.

Define $g(x) = f(x) - (\frac{f(b)-f(a)}{b-a})x$

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So there exists a point $c \in (a, b)$ such that $g'(c) = 0$.

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Thus

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$\square$

Compare auxillary function.

Proof.

Define $g(x) = f(x) - (\frac{f(b)-f(a)}{b-a})x$
Then observe that g satisfies the conditions of the Rolle's theorem.
So there exists a point $c \in (a, b)$ such that $g'(c) = 0$.
Thus

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$\square$

The auxillary function constructed in the proof of Lagrange's theorem can also be chosen as :

Try this function

$$F(x) = f(x) - [f(a) + \frac{f(b) - f(a)}{b - a}(x - a)]$$

$\Rightarrow$ F satisfies conditions of Rolle's thm & $F'(c) = 0$ gives us proof.

Theorem

Let f and g be continuous on $[a,b]$ and differentiable on (a,b). If $g'(x) \neq 0 \ \forall x \in (a,b)$, then there is a number $c \in (a,b)$ such that

$$\left[\frac{f(b)-f(a)}{g(b)-g(a)}\right] = \frac{f'(c)}{g'(c)}$$

/.

$h'(c) = 0$

Verify

$h(a) = h(b)$

$\Rightarrow h'(c) = 0$

Proof.

Apply Rolle's theorem to the function

Auxillary function.

$$h(x) = [f(b)-f(a)]g(x) - [g(b)-g(a)]f(x).$$

□

$R \Leftrightarrow L \Rightarrow C$

Pugh

Math. Analysis

Remarks

- Rolle's, Lagranges's and Cauchy's theorems are mutually equivalent

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- If we put $b = a + h$, then the Mean Value Theorem can be stated in the form:

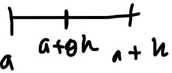
$$f(a+h) = f(a) + hf'(a+\theta h), \quad \underline{\underline{0 < \theta < 1}}$$

b

$f(a+h) - f(a)$

$= h f'(a+\theta h)$

$(b-a)$ c



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$f(x) = ax^2 + bx + c$

- Apply Mean Value Theorem to functions $1, x, x^2$ and check that in each case

LHS $\frac{y^2 - x^2}{y - x}$

$$\frac{f(y) - f(x)}{y - x} = \frac{f'(y) + f'(x)}{2}, \ \forall x, y \in \mathbb{R}$$

$\frac{2y + 2x}{2}$

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- Apply Mean Value Theorem to functions $1, x, x^2$ and check that in each case

$$\frac{f(y) - f(x)}{y - x} = \frac{f'(y) + f'(x)}{2}, \ \forall x, y \in \mathbb{R}$$

- The above equation holds for any quadratic polynomial and interestingly, the converse holds !

If f satisfies the equality then f is quadratic polynomial

$a \quad x_1 \quad x_2 \quad b$

$f(x_2)-f(x_1)$
$= f'(c)(x_2-x_1)$
$= 0$
$f(x_2) = f(x_1)$

- If f is differentiable on (a, b) such that $f'(x) = 0 \quad \forall x \in (a, b)$, then f is constant on (a, b).
- If f is differentiable on (a, b) such that $f'(x) > 0 \quad \forall x \in (a, b)$, then f is increasing on (a, b).
- If f is differentiable on (a, b) such that $f'(x) < 0 \quad \forall x \in (a, b)$, then f is decreasing on (a, b).
- If f is differentiable on (a, b) and there exist m, M such that $m \leq f'(x) \leq M \quad \forall x \in (a, b)$, then

$$m(b-a) \leq f(b) - f(a) \leq M(b-a)$$

$x_1 \quad x_2$

$f(x_2)-f(x_1)$
$= f'(c)(x_2-x_1)$
$\Rightarrow f(x_2) > f(x_1)$

Mean Value Inequality

$f(b) - f(a) = (b-a) f'(c)$

$m(b-a) \leq (b-a) f'(c) \leq (b-a) M$

Examples

1. Prove that $|\cos x - \cos y| \leq |x - y|, \ \forall x, y \in \mathbb{R}$
2. If $x > 0$, prove that $\frac{x}{1+x} < \log(1 + x) < x$
3. If f is continuous on $[0, 2]$ and twice differentiable on $(0, 2)$ and if If f(0) $= 0$; f(1) $= 1$ and f(2) $= 2$; then show that there exists x_0 such that $f^{(2)}(x_0) = 0$.

Apply MVT to $f(x) = \log x$ on $[1, 1+x]$, $x > 0$

$f(x) = \cos$

$y \quad x$

$$|\cos x - \cos y| = |-\sin(c)||(x-y)|$$

MVT

$$\leq |x - y|$$

$$\log(1+x) - \log 1 = \frac{1}{c}(x)$$

$$1 \leq c \leq 1+x$$

$$\frac{x}{1+x} \leq \frac{x}{c} \leq x$$

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4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is a decreasing function. If a, b, c are real numbers with $a < c < b$, prove that $(b - c)f(a) + (c - a)f(b) \leq (b - a)f(c)$.

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4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is a decreasing function. If a, b, c are real numbers with $a < c < b$, prove that $(b - c)f(a) + (c - a)f(b) \leq (b - a)f(c)$. ⇒ MVT on $[a, c]$ & $[c, b]$ & use f' decreasing
5. Let f be a function, continuous on $[a, b]$ and differentiable on (a, b). Let α be a real number. If $f(a) = f(b)$, then prove that there exists $x_0 \in (a, b)$ such that $\alpha f(x_0) + f'(x_0) = 0$.

construct $g(x) = e^{\alpha x} f(x)$

1. Suppose that f is differentiable on $\mathbb{R}$ and that $f(0) = 1, f(1) = 1$ and $f(2) = 1$. Show that (i) $f'(x) = \frac{1}{2}$ for some $x \in (0, 2)$. (ii) $f'(x) = \frac{1}{7}$ for some $x \in (0, 2)$.
2. Use MVT to prove that $\dfrac{27}{16} < \sqrt{3} < \dfrac{7}{4}$ and $\dfrac{20}{9} < \sqrt{5} < \dfrac{9}{4}$

   MVT to f^n $\sqrt{x}$
3. Show that the cubic $2x^3 + 3x^2 + 6x + 10$ has exactly one real root.
4. Find

$$\lim_{x\to\infty} x^2(tan^{-1}(x+1) - tan^{-1}(x))$$

MVT $\rightarrow$ $(\tan^{-1} x)' = \dfrac{1}{1+x^2}$

References

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